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11.9

Recall  \[ \frac{1}{1-x} = 1 + x + x^2 + \ldots \]

Similarly  \[ \frac{1}{1+x^2} = \frac{1}{1-(x^2)} = 1 - x^2 + x^4 - x^6 + \ldots \]

\[ = \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n} \]

\[ R = 1 \quad \text{Interval of convergence is } (-1, 1) \]
But \[ \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \]

**Theorem:** If \[ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \] has radius of convergence \( R \), then \( f(x) \) is differentiable and integrable on \( (a-R, a+R) \) and

\[ f'(x) = \sum_{n=1}^{\infty} n \cdot c_n (x-a)^{n-1} \]

\[ \int f(x) \, dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} + C \]

In other words, you can integrate and differentiate power series term by term!

This is cool!
Furthermore, the series for $f'(x)$ and $\int f(x) dx$ also have radii of convergence $R$. The behaviour at the end points may be different for the three series. \((x^2 < 1 \to |x| < 1)\)

Example: \[ \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots \]

\[ \int \frac{1}{1+x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + C \]

\[ \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + C \]

What should the constant $C$ be? Plug in $x=0$ to see $\arctan 0 = C = 0$
\[ \Rightarrow \ \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \pm \ldots \]

\[ R = 1 \quad \text{(inherited from the series for } \frac{1}{1+x^2} \text{)} \]

We need to consider the endpoints separately:

\[ x = 1 \quad \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \ldots \]

\[ \frac{\pi}{4} \quad \text{yes} \quad \text{converges by AST} \]

\[ \text{since } i) \lim_{n \to \infty} \frac{1}{2n+1} = 0 \]

\[ \text{and } \frac{1}{2n+3} < \frac{1}{2n+1} \text{ for all } n \]

\[ \text{Leibniz' formula} \]
\[ x = -1 \]

\[
\arctan(-1) = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots
\]

\[ = \frac{-\pi}{4} \]

Converges by AST for some reasons.

For the interval of convergence for Gregory's series is \([-1, 1] \).

Closed form for the series

\[
\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan(x).
\]

Example:

\[ \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots \quad R = 1 \]

Integrate to get
\[ \ln |1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + C \]

\[ x=0 \quad \ln 1 = C \]

\[ \Rightarrow C = 0. \]

\[ x \geq 0 \quad \ln (1+X) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \]

\[ R=1 \quad \text{as above.} \]

\[ x=1 \quad \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \quad \text{true! Alternating harmonic series!} \]

\[ x=-1 \quad -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \quad \text{diverges to -} \infty \quad \text{and in a sense this is reasonable since} \]
as \( x \to -1 \), \( \ln|1+x| \to -\infty \).

We now have series for \( \ln|1+x| \), 
\( \arctan x \), \( \frac{1}{1-x} \), and a bunch of others.

Why not for, say, \( e^x \), \( \sin x \), \( \sinh x \), ...

11.10 Taylor & McLaurin Series.

Assume for the moment that \( f(x) \) has a power series representation centered at \( x=a \), namely...
Assume \( f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \).

Need to find a formula for the coefficients \( c_n \) in terms of \( f(x) \).

\[
f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots
\]

\[
f(a) = c_0 + 0 + 0 + 0 + \ldots
\]

\[
\therefore \quad c_0 = f(a)
\]

\[
f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots
\]

\[
f'(a) = c_1 = \frac{f'(a)}{1}
\]

\[
f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \ldots
\]

\[
f''(a) = 2c_2 = \frac{f''(a)}{2}
\]
\(-q-\)

\[ f''''(a) = 3 \cdot 2 \cdot C_3 \]

\[ \Rightarrow C_3 = \frac{f''''(a)}{6} \]

\[ 6 = 2 \cdot 3 = 3! \]

\[ C_4 = \frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{f^{(4)}(a)}{4!} \]

\[ \Rightarrow C_n = \frac{f^{(n)}(a)}{n!} \]

If \( f \) is differentiable forever, then the series given by

\[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \]

is called the Taylor series of \( f \) centered at \( x=a \).
If $a = 0$, we call it the McLaurin series.

**Example:** Find the Taylor series for $f(x) = e^x$ centered at $a = 0$

$f'(x) = e^x \quad f'(0) = e^0 = 1$

$f''(x) = e^x \quad \text{same } f^{(n)}(0) = 1$

:. The Taylor/McLaurin series generated by $e^x$ is

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x
$$
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \]

\[ = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \ldots \]

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Keep studying!