Notes: The test is closed notes, closed book. Please put your cell phones and other electronic devices in airplane mode or turn them off. Make sure your writing is clear, concise, complete and, of course, correct.

Your Name: 

Problem 1: out of 15
Problem 2: out of 15
Problem 3: out of 20
Problem 4: out of 15
Problem 5: out of 15
Problem 6: out of 20
Total: out of 100

Good luck and have a great weekend!
1. (15 points)

(a) (5 points) Let $a$ be an integer. Prove the following proposition:
If $a^2$ is even, then $a$ is even.

(b) (10 points) Prove that $\sqrt{2}$ is irrational.
2. (15 points) Let $A, B$ and $C$ be sets.

(a) Prove or disprove: If $A \subseteq B$, then $A \times C \subseteq B \times C$

(b) Prove or disprove: If $A \times C \subseteq B \times C$, then $A \subseteq B$
3. (20 points) Let \( f_n \) be the \( n^{th} \) Fibonacci number. Recall that the Fibonacci numbers are defined recursively as follows:

\[
f_1 = f_2 = 1 \text{ and } f_{n+1} = f_n + f_{n-1} \text{ for all } n \geq 2
\]

Prove that for each natural number \( n \), \( f_n \) is even iff \( n \equiv 0 \pmod{3} \).
4. (15 points) A sequence $a_n$ is defined recursively as follows:

$$a_0 = 0, a_1 = 1, \text{ and for all } n \geq 2, a_n = 5a_{n-1} - 6a_{n-2}$$

Prove that for each natural number $n$,

$$a_n = 3^n - 2^n$$
5. (15 points) Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{R}^-$ the set of negative real numbers and $\mathcal{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers. Let

$$A_i = \{x \in \mathbb{R} \mid -\infty < x \leq i\} = (-\infty, i]$$

(a) Find $A_i$ for $i = -5, 0, \frac{1}{\pi}$

(b) Find $\bigcup_{i \in \mathcal{N}} A_i$

(c) Find $\bigcap_{i \in \mathcal{N}} A_i$

(d) Find $\bigcup_{i \in \mathbb{R}^-} A_i$

(e) Find $\bigcap_{i \in \mathbb{R}^+} A_i$
6. (20 points) Let $A$ and $B$ be two non-empty sets, and let $\mathcal{N}$ be the set of natural numbers.

(a) (5 points) Define what it means for a function $f : A \rightarrow B$ to be one-to-one.

(b) (5 points) Define what it means for $f : A \rightarrow B$ to be onto.

(c) (10 points) Consider the function $S : \mathcal{N} \rightarrow \mathcal{N}$ defined by

$$S(n) = \text{number of divisors of } n$$

i. Determine $S(n)$ for $n = 1, 2, 3, 4$ and $n = 24$.

$S(1) =$

$S(2) =$

$S(3) =$

$S(4) =$

$S(32) =$

ii. Is $S(n)$ one-to-one? Justify your answer.

iii. Is $S(n)$ onto? Justify your answer.