Notes: Show your work. In other words, just writing the answer, even if correct, may not be sufficient for full credit. Scientific calculators are allowed, but no programmable and/or graphing calculators. And please put away your cell phones and other electronic devices, turned off or in airplane mode.

Your Name: Solutions

Problem 1: out of 20
Problem 2: out of 20
Problem 3: out of 20
Problem 4: out of 20
Problem 5: out of 20
Total: out of 100

Good luck and have a great Memorial Day Weekend!
1. (20 points) In this problem, let the curve $C$ be the boundary of the square with vertices $(-1,0)$, $(0,1)$, $(1,0)$ and $(0,-1)$, oriented counterclockwise.

(a) (6 points) Evaluate

$$
\int_C ds = \text{length of the curve} = 4\sqrt{2}.
$$

(b) (7 points) Evaluate

$$
\int_C -y\,dx + x\,dy = \text{twice the area of the region (Green's Theorem)} = 4.
$$

(c) (7 points) Evaluate

$$
\int_C 2xy\,dx + x^2\,dy = 0
$$

Since if $f = x^2y$, then $\nabla f = (2xy, x^2)$ and thus the integral vanishes.
2. (20 points) Let \( \vec{F} = (-y, x, z) \). Evaluate

\[
\int_C \vec{F} \cdot d\vec{s}
\]

where the curve \( C \) is the quarter circle of radius \( \sqrt{2} \) centered at the origin, starting at the point \( P(1, 1, 0) \) and ending at the point \( Q(0, 0, \sqrt{2}) \).

Use \( \vec{C}(t) = (\cos t, \cos t, \sqrt{2} \sin t) \), \( 0 \leq t \leq \frac{\pi}{2} \)

to parametrize the curve.

This works since the curve can be described as the intersection of the sphere \( x^2 + y^2 + z^2 = 2 \)

with the plane \( y = x \), and \( \vec{C}(t) \) checks both equations and preserves orientation.

Now \( \vec{C}'(t) = (-\sin t, -\sin t, \sqrt{2} \cos t) \) and

\[
\int_C \vec{F} \cdot \vec{d}s = \int_0^{\pi/2} \vec{F} \cdot \vec{C}'(t) dt = 0
\]

\[
= \int_0^{\pi/2} \sin t \cos t - \sin t \cos t + 2 \sin t \cos t dt
\]

\[
= \int_0^{\pi/2} \sin 2t dt = -\frac{1}{2} \cos 2t \bigg|_0^{\pi/2} = \frac{1}{2} + \frac{1}{2} = 1
\]
3. (20 points) Let \( \Phi(u, v) = (u \cos v, u \sin v, u^2) \) be a mapping from a subset \( D \subseteq \mathbb{R}^2 \) given by \( 0 \leq u \leq 3, 0 \leq v \leq 2\pi \) onto a surface \( S = \Phi(D) \) in \( \mathbb{R}^3 \).

(a) (5 points) Sketch or describe (identify) this surface.

\[ z = x^2 + y^2 \quad \text{where} \quad z \leq 3 \quad \text{(and} \quad z > 0 \text{).} \]

(b) (15 points) Find the surface area of \( S \).

\[ \vec{T}_u = ( \cos v, \sin v, 2u) \]
\[ \vec{T}_v = (-u \sin v, u \cos v, 0) \]

\[ \vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (-2u^2 \cos v, 2u^2 \sin v, u) \]

\[ ||\vec{T}_u \times \vec{T}_v|| = \sqrt{4u^4 \cos^2 v + 4u^4 \sin^2 v + u^2} = \sqrt{u^2 (4u^2 + 1)} = u \sqrt{4u^2 + 1} \]

\[ A(S) = \iint_S dS = \iint_{D} ||\vec{T}_u \times \vec{T}_v|| \, dudv = \iint_{0}^{2\pi} \iint_{0}^{3} u \sqrt{4u^2 + 1} \, du \, dv \]

\[ = \frac{1}{8} \int_{0}^{2\pi} \int_{0}^{3} 8u \sqrt{4u^2 + 1} \, du \]

\[ = \frac{2\pi}{8} \cdot \left[ \frac{2}{3} (4u^2 + 1)^{3/2} \right]_{0}^{3} \]

\[ = \frac{2\pi}{6} \left( \frac{2}{3} 3^{3/2} - \frac{2}{3} \right) = \frac{\pi}{6} (3^{3/2} - 1) \]
4. (20 points) Let the surface \( S \) be the portion of the cone \( z^2 = x^2 + y^2 \) where \( 1 \leq z \leq 4 \). Let \( f(x, y, z) = y + z^3 \). Evaluate

\[
\iiint_S f \, dS
\]

Use \( \Phi(u, v) = (u \cos v, u \sin v, u) \) with \( 0 \leq v \leq 2\pi, \ 1 \leq u \leq 4 \), to parameterize \( S \).

\[
\begin{align*}
\mathbf{T}_u &= (\cos v, \sin v, 1) \\
\mathbf{T}_v &= (-u \sin v, u \cos v, 0)
\end{align*}
\]

\[
\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos v & \sin v & 1 \\
-u \sin v & u \cos v & 0
\end{vmatrix} = (-u \cos v, -u \sin v, u)
\]

\[
|\mathbf{T}_u \times \mathbf{T}_v| = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v + u^2} = \sqrt{2u^2} = u\sqrt{2}
\]

\[
\int_0^{2\pi} \int_1^4 f(u, v) |\mathbf{T}_u \times \mathbf{T}_v| \, du \, dv = \int_0^{2\pi} \int_1^4 (u \sin v + u^3) u \sqrt{2} \, du \, dv
\]

\[
= \sqrt{2} \int_0^{2\pi} \left[ \frac{u^4}{4} \right]_1^4 + \frac{u^4}{5} \left[ \frac{u^5}{5} \right]_1^4 \, dv
\]

\[
= \sqrt{2} \int_0^{2\pi} \left( \frac{256}{4} - \frac{1}{4} \right) + \frac{1024}{5} \left( \frac{1024}{5} - \frac{1}{5} \right) \, dv
\]

\[
= \frac{2\sqrt{2}}{5} [1023](11)
\]
5. (20 points) Let \( \vec{F} = (x, -z, y) \) and \( S \) be the closed upper hemisphere centered at the origin of radius 3, with the orientation given by the outer normal (i.e. pointing out of the hemisphere). In other words, \( S \) is the surface enclosed by the portion of the sphere \( x^2 + y^2 + z^2 = 9 \) where \( z \geq 0 \) and the disk \( x^2 + y^2 \leq 9 \) where \( z = 0 \). Evaluate

\[
\iint_S \vec{F} \cdot d\vec{S}
\]

(a) \( \iint_{S_1} \vec{F} \cdot d\vec{S} = \frac{1}{3} \iint_S (x - z, y) \cdot (x, y, z) \, dS = \frac{1}{3} \iint_{S_1} x^2 - z^2 + y \, dS \)

\[
= \frac{1}{3} \iint_{S_1} x^2 \, dS. \text{ Using symmetry we get } x^2 + y^2 + z^2 \text{ on the whole sphere}
\]

\[
= \frac{1}{3} \iint_{S_1} x^2 \, dS = \frac{1}{3} \iint \frac{x^2}{R^2} \, dS = \frac{1}{3} \iint \frac{R^2}{9} \, dS = \frac{1}{9} \iint R^2 \, dS = \frac{4}{9} R^2 = \frac{4}{9} \cdot 9 = \frac{36}{9} \lll \frac{18\pi}{3}
\]

(b) \( \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_S (x - z, y) \cdot (0, 0, -1) \, dS = \iint_{S_2} -y \, dS \)

\[
= \int \int_{S_2} -r \sin \theta \, r \, dr \, d\theta = \int_0^{2\pi} \sin \theta \, d\theta \int_0^3 r^2 \, dr = 0
\]

\[
\iint_S \vec{F} \cdot d\vec{S} = \frac{18\pi}{3}
\]