Problem 8: For the sake of a contradiction, let
\[ A = \{ y \mid y = \{ x \} \text{ for some set } x \} \]
be a set. Then using the pairing axiom with \( A \) and \( A \), \( \{ A, A \} = \{ A \} \) is also a set. Then \( U \{ A \} = \{ x \mid y \in A \} = \{ x \mid x \text{ is a set} \} \)
but this is not a set. \( \square \).

Problem 9: Let \( A = \{ \emptyset \} \) \( B = \{ \{ \emptyset \} \} \). So \( A \in B \). But
\[ Pa = \{ \emptyset, \{ \emptyset \} \} \text{ and } PB = \{ \{ \{ \emptyset \} \}, \emptyset \} \]
and clearly \( Pa \notin PB \).
#7) This is done by induction. Assume \( h_1 \) and \( h_2 \) are functions that satisfy the conclusion of the recursion theorem. Let \( S = \{ n \in \mathbb{N} \mid h_1(n) = h_2(n) \} \). It suffices to show \( S \) is inductive. Since both \( h_1 \) and \( h_2 \) are acceptable, \( 0 \in S \).

Now assume \( n \in S \) and consider:

\[
\begin{align*}
    h_1(n^+) &= F(h_1(n)) \quad \text{since } h_1(n) = h_2(n) \\
    h_2(n^+) &= F(h_2(n))
\end{align*}
\]

It follows that \( h_1(n^+) = h_2(n^+) \), so \( n^+ \in S \). \( \square \)

#8) The idea here is that if \( h(n^+) = h(m^+) \) and \( n^+ \neq m^+ \), then \( f^{(m)}(0^+) = f^{(m)}(0^+) \) and if \( m < n \) say, \( f^{(n-m)}(0) = 0 \), which contradicts \( c \notin \text{ran}(f) \).

Double since \( f \) is one-to-one.

To do this in our setup, we use induction.

Let \( S = \{ n \in \mathbb{N} \mid \forall m \in \mathbb{N}, \ h(n) = h(m) \rightarrow n = m \} \).

B.C. \( 0 \in S \).

Assume \( 0 \neq 0 \). Then \( m = \rho^+ \) for some \( \rho \).

So \( h(\rho^+) = h(0) \) but \( h(\rho^+) = f(h(\rho)) \) so \( c \notin \text{ran}(f) \) \( \uparrow \)

Therefore \( 0 \in S \).
Ind. Step: Assume \( n \in S \), show \( n^+ \in S \).

Again assume \( h(n^+) = h(m) \). By the above

\[ m \neq 0 \text{ so } m = p^+ \text{ for some } p. \]

So

\[ h(n^+) = h(p^+) \]

So

\[ f(h(n)) = f(h(p)) \]

Since \( f \) is 1-1

\[ h(n) = h(p) \]

Since \( n \in S \), \( n = p \) and hence \( n^+ = p^+ \).

So

\[ n^+ \in S \]

\[ \square \]

Problem 13:

Assume \( m \cdot n = 0 \) show \( m = 0 \) or \( n = 0 \).

Contrapositive: Assume \( m \neq 0 \), \( n \neq 0 \) show \( m \cdot n \neq 0 \).

\[ m \neq 0 \implies k^+ = m^+ \text{ for some } k \in \mathbb{W}. \]

\[ n \neq 0 \implies l^+ = n^+ \text{ for some } l \in \mathbb{W}. \]

\[ l^+ \cdot k^+ = \frac{M_2}{A_2} \]

\[ \Delta_2 \]

\[ (l^+ \cdot k + l)^+ \]

\[ \neq 0 \]

Since \( 0 \neq r^+ \) for any \( r \) and

\[ l^+ \cdot k + l \in \mathbb{W} \]

Since \( 0^+ \), \( 1^+ \) are total functions.
Problem 14:

Set \( S = \{ n \in \mathbb{N} \mid n \) is either even or odd but not both \( 3 \}. \)

Base Case: \( n = 0 \)

\[ 0 = 2 \cdot 0 \] by M1, so \( 0 \) is even.

Assume that \( 0 = 2k + 1 \)

\[ 2k = 2k + 0 \]

\[ = (2k+0)^+ \]

so \( 0 \) is not odd, so \( 0 \in S. \)

Induction Step: Assume \( n \) is either even or odd but not both. Show same for \( n^+ \).

Case 1: \( n \) is even.

\[ n = 2 \cdot k \]

\[ \Rightarrow n^+ = (2k)^+ \]

\[ = (2k+0)^+ \]

\[ = 2k + 0^+ \]

\[ = 2k + 1 \]

so \( n^+ \) is odd

Case 2: \( n \) is odd

\[ n = 2k + 1 \]

\[ n^+ = (2k+1)^+ \]

\[ = 2k + 1^+ \]

\[ = 2k + 2 \]

\[ = 2(k+1) \]

\[ \Rightarrow = 2k^+ \]

so \( n^+ \) is even.
14 cont'd:
Now if \( n^+ \) is both even and odd, it is
done that the above steps can be reversed to show
that \( n \) is both even and odd, which contradicts \( n \in S \). Therefore \( n^+ \in S \), \( S \) is inductive.

Problem 20: Clearly if \( 0 \notin A \) and \( A \neq \emptyset \), then since \( 0 \in m^+ \) in \( 0 \in cUA \) and \( A \neq \emptyset \).

Now let \( \tilde{n} \) be the least \( m \in A \) s.t. \( n^+ \notin A \). Such an
it must exist unless \( A = \omega \) since \( A \neq \emptyset \). (assuming \( 0 \in A \))

Assume \( A \neq \omega \) and consider this \( \tilde{n} \):

Case 1: \( \tilde{n} \) is the largest \( m \in A \).
Then \( \tilde{n} \in A \) but \( UA = \{ m \mid m \in \tilde{n} \} \)
so \( UA \neq A \).

Case 2: \( \tilde{n} \) is not the largest. But \( \tilde{n}^+ \notin A \), so let
\( m \in A \) s.t. \( m \leq \tilde{n} \).
Clearly \( \tilde{n}^+ \in UA \) but \( \tilde{n}^+ \notin A \) so
in either case \( UA \neq A \).

Now if \( A = \omega \), then since \( A \) is transitive, \( UA \leq A \).
But given \( m \in \omega \), \( m \in m^+ \in A \), so \( m \in UA \).
Therefore \( U \omega = \omega \).
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Problem 26: By induction: \[ S = \{ n \in \mathbb{N} \mid \forall f: n^+ \to \omega \text{ has a largest} \} \]
\[ n = 0 \text{ if } 0 \to n \text{ largest is } f(0) \text{ since it's the sole member of ran } f \]
\[ \text{Consider } f \cap n^+ \text{ By I.H. } f \cap n^+ \text{ has largest output, say } q \in \omega. \]
\[ \text{Case 1: } f(n^+) \leq q. \text{ Then } q \text{ is largest in ran } f \]
\[ 2: \quad f(n^+) > q. \text{ Then } f(n^+) \text{ is largest in ran } f. \]
By trichotomy these are the only cases to consider. \( \square \).
So \( n^+ \in S, S \) is inductive.

Problem 32:
(a) \( A = \{ 1 \} \quad A^+ = \{ 1 \} \cup \{ 1, 1 \} = \{ 1, 1 \} \quad (b) \ U (\{ 23^+ \}) \]
\[ U A^+ = \{ 0 \} = \{ 0, 1, 2 \} \]

Problem 33:
(a) Transitive
(b) not transitive, 0 is missing
(c) \( 0 \cup 1 = \{ 0, 1 \} \quad 0 \) is missing

Problem 34:
(a) \( a = \emptyset, b = \emptyset \) will do
(b) \( c = \{ \emptyset \}, d = \{ \emptyset \}, e = \emptyset \) will do.