Problem 1:
Let \( A_n = \left\{ -n^3, n \left( \frac{1}{n}, \frac{3n^2 - 1}{n} \right) \right\} \). Find \( \bigcap_{\text{new}} A_n \) and \( \bigcup_{\text{new}} A_n \). Justify your answers.

Problem 2: Given a family \( \{ R_t \}_{t \in T} \) of equivalence relations in a set \( X \), check if
\( R_1 = \bigcap_{t \in T} R_t \) is an equivalence relation.
\( R_2 = \bigcup_{t \in T} R_t \) is an equivalence relation.

Problem 3: Let \( a > 0 \). Show that \( \sum_{k=1}^{n} \frac{1}{(a+k-1)(a+k)} = \frac{n}{a(a+n)} \).

Problem 4: Find and prove a formula for
\( \frac{1}{(i(i+1))} \).

Problem 5: Let \( f : A \to A \) be defined by \( f(a) = a^2 + 1 \).
(a) If \( A = \mathbb{N} \), is \( f \) a surjection? injection?
(b) If \( A = \mathbb{Z} \), is \( f \) a surjection? injection?

Problem 6: Let \( g : A \to B \) and \( f : B \to C \). Prove that if \( f \) and \( g \) are one-to-one, then \( f \circ g \) is one-to-one. Find an example of \( f \) and \( g \) such that \( f \circ g \) is one-to-one, but \( f \) is not one-to-one.

Problem 7: Show that \( \frac{(2n)!}{(n!)^2} \) is an even integer for all \( n \in \mathbb{N} \).
Problem 8: Give a combinatorial (sic a counting argument) and a direct proof of
\[ C(n,k) = C(n, n-k) \quad \text{or} \quad \binom{n}{k} = \binom{n}{n-k} \]
where \( k, n \in \mathbb{N}, \ 0 \leq k \leq n \).

Problem 9: Let \( f_i : A_i \rightarrow \mathbb{Z} \) be defined by
\[ f_i(x) = \prod_{j \in I_i} (x - j) \] for which \( i \in \{1, 2, 3\} \)
As \( f_1(x) = x_{A_1}(x) \) given that \( A_1 = \mathbb{R}, A_2 = \mathbb{R} - \mathbb{N}, A_3 = \mathbb{R} - \mathbb{Z} \).
For more \( i \) where \( f_i(x) \neq x_{A_i}(x) \) find \( B_i \) such that \( f_i(x) = x_{B_i}(x) \).

Problem 10: Use the Schroeder-Bernstein Theorem to show that \((0,1)\) and \(\mathbb{R}\) have the same cardinality.

Problem 11: Let \( n \in \mathbb{N} \), \( n \) not a prime. Show that
\[ (n-1)! \equiv 0 \pmod{n} \quad \text{for} \ n \neq 4 \]

Problem 12: Let \( f : \mathbb{N}^2 \rightarrow \mathbb{N} \) be defined by
\[ f(x,y) = x^2 + y^2 . \]
Find \( f^{-1}(405), f^{-1}(193), f^{-1}(253) \).

Problem 13: What is the cardinality of \( \{ x \in \mathbb{R} : \exists n \in \mathbb{N} (x^n \in \mathbb{Z}) \} ? \)

Problem 14: Let \( P \) be the set of all formulas of the propositional calculus. Define a relation \( \sim \) on formulas \( \Phi \) from \( P \) as follows
\[ \Phi_1 \sim \Phi_2 \iff \Phi_1 \equiv \Phi_2 \text{ is a tautology} . \]
Problem 14 continued
(a) Prove that \( \sim \) is an equivalence.
(b) What formulas belong to \([AJ]\) where
\( A \) is the formula \( \neg (p \land q) \)
(c) What formulas belong to \([BJ]\) where
\( B \) is the formula \( \neg (p \lor q) \)

Problem 15: Prove that the mapping
\( f: \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \)
defined by \( f(x,y) = 2^x (2y+1)-1 \) is a bijection.
Conclude that \( \text{card} (\mathbb{N}_0 \times \mathbb{N}_0) = \text{card} (\mathbb{N}) \).

Problem 16: Construct a mapping \( f: (0,1) \to \mathbb{N} \) such that
\( f \) is onto and \( \forall n \in \mathbb{N} \) \( \text{card} (f^{-1}(\{n\})) = \text{card}(\mathbb{R}) \).
In other words, construct \( f \) so that the pre-image of any natural number has the cardinality of the continuum.

Problem 17: As \#16, but construct \( f \) so that the pre-image of an even integer has the cardinality of the continuum and the pre-image of an odd integer has cardinality 1.

Problem 18: Prove or disprove. Let \( A, B \) be sets
(a) \( A \cup (A \cap B) = A \)  (b) \( A \cap (A \cup B) = B \)

Problem 19: Prove that \( \forall \in \mathbb{N} \)
\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

Problem 20: Is \( (p \rightarrow q) \leftrightarrow [(p \land q) \leftrightarrow p] \) a tautology?