Notes: The test is closed notes, closed book. Please put your cell phones and other electronic devices in airplane mode or turn them off. Make sure your writing is clear, concise, complete and, of course, correct.

Your Name: 

Solutions

Problem 1: out of 10
Problem 2: out of 20
Problem 3: out of 20
Problem 4: out of 15
Problem 5: out of 15
Problem 6: out of 20
Total: out of 100

Good luck and have a great weekend!
1. (10 points) Let $x$ be a positive real number. Prove or disprove the following propositions:

(a) If $x$ is irrational, then $x^2$ is irrational.

False. Let $x = \sqrt{2}$ (irrational, we proved $x$ is in class).

$$x^2 = (\sqrt{2})^2 = 2$$ which is rational.

(b) If $x$ is irrational, then $\sqrt{x}$ is irrational.

True. Prove this by contrapositive:
Assume $\sqrt{x}$ is rational. Show $x$ is rational.
Since $\sqrt{x}$ is rational, there exist integers $m, n, n \neq 0$ so that
$$\sqrt{x} = \frac{m}{n}$$

$$x = \frac{m^2}{n^2}$$

and since $\mathbb{Z}$ is closed under multiplication, and $n \neq 0$, $x$ is rational.
2. (20 points)

(a) Let $a$ be an integer. Prove the following proposition:
If $a^2 \equiv 0 \pmod{5}$, then $a \equiv 0 \pmod{5}$.

Proof by contrapositive:

If $a \not\equiv 0 \pmod{5}$, then $a^2 \not\equiv 0 \pmod{5}$.

Do this by cases:

Case 1): $a \equiv 1 \pmod{5} \Rightarrow a^2 \equiv 1 \pmod{5}$

Case 2): $a \equiv 2 \pmod{5} \Rightarrow a^2 \equiv 4 \pmod{5}$

Case 3): $a \equiv 3 \pmod{5} \Rightarrow a^2 \equiv 4 \pmod{5}$

Case 4): $a \equiv 4 \pmod{5} \Rightarrow a^2 \equiv 1 \pmod{5}$

Since these four cases are exhaustive (by the division algorithm), the result follows.

(b) Prove that $\sqrt{5}$ is irrational.

Proof by contradiction:

Assume $\sqrt{5}$ is rational. Hence there exist $m, n, n \neq 0$ such that $\sqrt{5} = \frac{m}{n}$. Furthermore, you can assume w.l.o.g. that $\frac{m}{n}$ is in reduced form, i.e. $m$ and $n$ have no common factors. Now $\sqrt{5} = \frac{m}{n}$ implies

$5 = \frac{m^2}{n^2} \Rightarrow 5n^2 = m^2 \Rightarrow m^2 \equiv 0 \pmod{5}$

and hence by (a), $m \equiv 0 \pmod{5}$. Therefore $m = 5k$ for $k \in \mathbb{Z}$. Thus $n^2 = 5k^2$ and by a similar argument you see that $n = 5\lambda$ for some $\lambda \in \mathbb{Z}$. Hence $m$ and $n$ have a common factor for a $\lambda$. \[\square\]
3. (20 points) Let $A$ and $B$ be sets. Recall that $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ is the power set of $A$.

(a) Prove or disprove:

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

Let $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$

$$= C \in \mathcal{P}(A) \land C \in \mathcal{P}(B)$$

$$= C \subseteq A \land C \subseteq B$$

$$= C \subseteq A \cap B$$

$$= C \in \mathcal{P}(A \cap B)$$.

Therefore, the above steps are true.

(b) Prove or disprove:

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$$

False. Let $A = \{1, 2\}$, $B = \{2, 3\}$.

Then $\mathcal{P}(A) \cup \mathcal{P}(B)$ cannot contain $\{1, 3\}$ because $3 \notin A$ and $1 \notin B$, but $\mathcal{P}(A \cup B)$ surely must contain $\{1, 3\}$ since $\{1, 3\} \subseteq A \cup B$.

\[\square\]
4. (15 points) Prove that for each natural number \( n \),

\[
\sum_{i=1}^{n} (4i - 3) = n(2n + 1)
\]

\textbf{Proof:} By induction on \( n \):

**Base Case** \( n = 1 \)

\[
\text{L.H.S. } 4 - 3 = 1
\]

\[
\text{R.H.S. } 1 \cdot (2 - 1) = 1
\]

**Induction Step** \( n \to n+1 \)

\[
\sum_{i=1}^{n+1} (4i - 3) = \sum_{i=1}^{n} (4i - 3) + 4(n+1) - 3
\]

I.H. \( \sum_{i=1}^{n} (4i - 3) = n(2n-1) + 4n + 4 - 3 \)

\[
= 2n^2 - n + 4n + 1
\]

\[
= 2n^2 + 3n + 1
\]

\[
= (n+1)(2n+1)
\]

\[
= (n+1) \cdot 2(n+1) - 1
\]

\( \square \)
5. (15 points) Prove that for each natural number \( n \),

Proof: By induction, \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \)

- Base Case: \( n = 1 \)

L.H.S.: \( \sum_{i=1}^{1} \frac{1}{i^2} = 1 \) and \( 1 \leq 1 \) ✓

R.H.S.: \( 2 - 1 = 1 \)

- Induction Step: \( n \Rightarrow n+1 \)

\[
\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}
\]

\[
\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}
\]

\[
= 2 - \frac{(n+1)^2 - n}{n \cdot (n+1)^2}
\]

\[
= 2 - \frac{n^2 + 2n + 1 - n}{n \cdot (n+1)^2}
\]

\[
= 2 - \frac{n^2 + n}{n \cdot (n+1)^2} + \frac{1}{n \cdot (n+1)^2}
\]

\[
> 2 - \frac{1}{n+1}
\]
6. (20 points) Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers and $\mathbb{N} = \{1, 2, 3, \ldots \}$. Let

$$A_t = \{ x \in \mathbb{R} \mid 1 - \frac{1}{t} < x \leq 1 + \frac{t}{t+1} \}$$

(a) Find $A_t$ for $t = 1, 2, \frac{1}{2}$

$A_1 = (0, \frac{3}{2}]$ \hspace{1cm} $A_2 = (\frac{1}{2}, \frac{5}{3}]$

$A_{\frac{1}{2}} = (-1, \frac{4}{3}]$

(b) Find $\bigcup_{t \in \mathbb{N}} A_t$

$$\bigcup_{t \in \mathbb{N}} A_t = (0, 2)$$

(c) Find $\bigcap_{t \in \mathbb{N}} A_t$

$$\bigcap_{t \in \mathbb{N}} A_t = [1, \frac{3}{2}]$$

(d) Find $\bigcup_{t \in \mathbb{R}^+} A_t$

$$\bigcup_{t \in \mathbb{R}^+} A_t = (-\infty, 2)$$

(e) Find $\bigcap_{t \in \mathbb{R}^+} A_t$

$$\bigcap_{t \in \mathbb{R}^+} A_t = \{1\}$$