The Irrationality of $\sqrt{2}$

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Abstract

In this paper we will show the irrationality of $\sqrt{2}$ for all natural numbers $n \geq 2$.

1 Introduction

The discovery that something as simple to construct as the diagonal of a square with side-length one is not expressible as a ratio of lengths based on a common unit length was a major blow to the philosophy of the Pythagorean School. Rumor even has it that the discoverer of this fact was killed to hide the truth ([1]). Nonetheless, it is difficult to hide a mathematical truth for long, and it is now common knowledge that many widely used numbers such as $\sqrt{2}$, $\pi$ and $e$ are irrational numbers. Our goal in this paper is to give the classical proof by contradiction to show that $\sqrt{2}$ is irrational and generalize this to $\sqrt{n}$.

2 Preliminaries: Definitions and Lemmas

In this section we state the main definitions and assumptions we will use throughout this paper. We will also prove a supporting lemma that we will need in the proofs of the main theorems. We will assume as given the sets of natural numbers, integers and real numbers. Since the property of being rational or irrational is central to the theorems in this paper, we will define them first.

**Definition 2.1.** Let $x$ be a real number. We say that $x$ is rational, provided there exists integers $p$ and $q$ with $q \neq 0$ such that $x = \frac{p}{q}$. Furthermore, we say that $x$ is irrational in case $x$ is not rational.

Note that the definition of irrational depends on the proper definition of real number, which can be found in [2].

**Definition 2.2.** Let $n$ be an integer. We say that $n$ is even, provided there exists an integer $m$ such that $n = 2m$. 
Definition 2.3. Let \( n \) be an integer. We say that \( n \) is odd, provided there exists an integer \( m \) such that \( n = 2m + 1 \).

We will assume without proof that every natural number is either even or odd but not both. Next we will prove a couple of facts about even and odd integers.

Lemma 2.4. Let \( n \) be a natural number. Then if \( n^2 \) is even, \( n \) is even.

Proof. We will prove this by proving the contrapositive. To do so, assume that \( n \) is an odd natural number. Therefore there exists an integer \( m \) so that \( n = 2m + 1 \). Hence

\[
\begin{align*}
n^2 &= (2m + 1)^2 \\
&= 4m^2 + 4m + 1 \\
&= 2(2m^2 + 2m) + 1 \\
&= 2k + 1
\end{align*}
\]

where \( k = 2m^2 + 2m \) is an integer by the closure property of the set of integers under addition and multiplication. This shows that \( n^2 \) is odd. \( \square \)

Before we will generalize the above result, we will prove the following lemma:

Lemma 2.5. Let \( n \) and \( m \) be odd integers. Then their product \( nm \) is odd also.

Proof. Assume that \( n \) and \( m \) are odd integers. Therefore there exist integers \( k \) and \( l \) so that \( n = 2k + 1 \) and \( m = 2l + 1 \). Hence

\[
\begin{align*}
nm &= (2k + 1)(2l + 1) \\
&= 4kl + 2k + 2l + 1 \\
&= 2(2kl + k + l) + 1 \\
&= 2p + 1
\end{align*}
\]

where \( p = 2kl + k + l \) is an integer, again by the closure property of the set of integers under addition and multiplication. This shows that the product of two odd integers is odd. \( \square \)

Lemma 2.6. Let \( n \) and \( k \) be natural numbers where \( k \geq 2 \). Then if \( n^k \) is even, then \( n \) is even.

Proof. We will again prove the contrapositive. Hence we will assume that \( n \) is odd and prove that \( n^k \) is odd for all \( k \geq 2 \). We do this by induction on the exponent \( k \). Clearly the base case \( k = 2 \) is given in the proof of Lemma 2.4. To prove the induction step, we assume that \( n^k \) is odd and observe that

\[
n^{k+1} = n^k n
\]

which therefore, as the product of two odd numbers, is odd by Lemma 2.5. \( \square \)
3 Main Results

In this section we will prove our main result, namely that $\sqrt{2}$ is irrational for all natural numbers $n \geq 2$. Before we prove the result in general, we will exhibit the ideas of the proof in the special case when $n = 2$.

**Theorem 3.1.** $\sqrt{2}$ is irrational.

**Proof.** We will prove this by contradiction. Hence, for the sake of a contradiction, assume that $\sqrt{2}$ is rational. Therefore there exist integers $p$ and $q$ with $q \neq 0$ so that $\sqrt{2} = \frac{p}{q}$. Without loss of generality we can assume that $p$ and $q$ have no common factors other than one, for if they did, using unique prime factorization of integers, we could factor $p$ and $q$ and cancel common factors to obtain integers $p'$ and $q'$ so that $\sqrt{2} = \frac{p}{q} = \frac{p'}{q'}$ where $p'$ and $q'$ have no common factors other than one. Thus

$$\sqrt{2} = \frac{p}{q}$$

$$\implies 2 = \frac{p^2}{q^2}$$

$$\implies 2q^2 = p^2$$

By Definition 2.2 this means that $p^2$ is even, and thus by Lemma 2.4 it follows that $p$ is even. Therefore there exists an integer $k$ so that $p = 2k$ and we get

$$2q^2 = (2k)^2$$

$$\implies 2q^2 = 4k^2$$

$$\implies q^2 = 2k^2$$

Again we deduce that $q^2$ is even, and by Lemma 2.4, that $q$ is even also. Therefore both $p$ and $q$ are even and thus share a common factor of two, which contradicts the assumption that $p$ and $q$ have no common factors. \qed

We will now generalize the above proof to prove the main result.

**Theorem 3.2.** For every natural number $n \geq 2$, $\sqrt{n} \sqrt{2}$ is irrational.

**Proof.** The proof is analogous to the proof above. Again, for the sake of a contradiction, assume that $\sqrt{2}$ is rational. Hence let $\sqrt{2} = \frac{p}{q}$ where $p$ and $q$ are non-zero integers with no common factors other than one. Thus

$$\sqrt{2} = \frac{p}{q}$$

$$\implies 2 = \frac{p^n}{q^n}$$

$$\implies 2q^n = p^n$$
Therefore \( p^n \) is even, and thus, by Lemma 2.6, it follows that \( p \) is even. Therefore there exists an integer \( k \) so that \( p = 2k \) and we get

\[
2q^n = (2k)^n \\
\implies 2q^n = 2^n k^n \\
\implies q^n = 2^{n-1} k^n \\
\implies q^n = 2(2^{n-2} k^n) \\
\implies q^n = 2m
\]

Since \( n \geq 2 \), \( m = (2^{n-2} k^n) \) is an integer, and it follows that \( q^n \) is even. Thus \( q \) is even also by Lemma 2.4. Once again we have a contradiction. \( \square \)

4 Closing Remarks

While the above proof may not seem too difficult, it was a major and disturbing discovery in its time. This discovery irrevocably changed the way mathematicians think about number and ratios. It is the mathematical equivalent of the fact that the earth is round rather than flat; things are just not as simple as they seem at first sight. The above proof is the classical proof by contradiction of this fact, but many different proofs exist. See [1] for some additional proofs and of the irrationality of \( \sqrt{2} \) and insights into the thoughts and philosophy of Mathematics of the Pythagoreans.

However, it is not at all trivial to prove that certain other numbers such as \( \pi \) or \( e \) are irrational. It is actually much easier to prove the more surprising fact that in some sense almost all real numbers must be irrational. To explain this, consider the following: if one were to take a dart and throw it at the real number line, the probability of hitting a rational number would be zero, and the probability of hitting an irrational would be one.

References
